A FINITELY ADDITIVE STRONG LAW

by

Roger A. Purves and William D. Sudderth*

University of California, Berkeley

University of Minnesota, Minneapolis

Technical Report No. 184

October 1972

University of Minnesota

Minneapolis, Minnesota

*Research partially supported by NSF Grant GP-24183 and by U.S. Army Research Grant DA-ARO-D-31-124-70-G102.

1. Introduction.

Let γ be a finitely additive probability defined on all subsets of the set N of positive integers. (The main body of the paper will consider probabilities on an arbitrary set.) If γ is countably additive, it is well known that there exists a unique countably additive probability which assigns to each subset of $N^N (= N \times N \times \ldots)$ of the form

$$\{z \in N^N | z_i \in A_i,\ i = 1,\ldots, j\},\quad j \in N,\ A_i \subseteq N,\ 1 \leq i \leq j,$$

the probability $\prod_{i=1}^{j} \gamma(A_i)$, and whose domain is the sigma-algebra generated by these sets. Is there a counterpart to this product measure theorem in the case that γ is not countably additive? In the first place, it is easy to see what probabilities should be assigned to the subsets of N^N which depend on only finitely many coordinates. However, once there, the conventional methods, relying as they do on the countable additivity of γ, give no indication as to what the values of the measure should be on a wider class of sets. In much greater generality this problem has already been considered by Lester Dubins and the late Leonard Savage in their book [2]. In order to surmount the apparent arbitrariness involved in the extension, Dubins and Savage require that a certain natural condition be satisfied, which for the special case being considered here, reduces to the following:

$$(1.1) \qquad \pi(D) = \int_N \pi(Dx)d\gamma(x).$$

Here π is the extension-to-be, $D \subseteq N^N$, $Dx = \{z \in N^N | (x, z_1, z_2, \ldots) \in D\}$. For reasons given in [2, pp. 12-20] but too lengthy to present here, it is natural to ask that (1.1) holds for all sets D which are clopen (simultaneously closed and open) in the product topology on N^N determined by assigning N

the discrete topology. Then, although Dubins and Savage do not do this directly, their method can easily be adapted to show that there is exactly one finitely additive probability π which is defined on the clopen subsets of N^N and which satisfies (1.1) for all clopen D.

To compare this situation with the countably additive one described in the opening paragraph, note that the collection of clopen sets in N^N includes properly the collection of sets which depend on finitely many coordinates. However, the clopen sets form a much smaller class than the domain of the conventional product measure. In fact, the latter coincides with the sigma-algebra generated by the former. This is the point of departure of our efforts. Is it possible to extend π in some natural way to a larger collection of sets? One time-honored way to proceed in such a situation takes place in two stages. First take the measure of each open set to be the supremum of the measures of the clopen sets contained within it, and the measure of each closed set to be one minus the measure of its complement. Second, form the collection of all sets which can be approximated from without by an open set and within by a closed set in such a way that the difference of their measures can be made arbitrarily small. It is clear how the extension should be defined on these sets.

It is not difficult to show that all this works out to give an extension of π to an algebra of sets containing the open sets (Theorem 2.1). The next question is: how large is this algebra? For example, does it contain all sets which are countable intersections of open sets? We found this question difficult, even in such a special case, and the results to follow come from our attempt to answer it. Our answer, in rather greater generality, (see Theorem 8.1 below) is yes. Perhaps of more probabilistic interest, the methods used in establishing Theorem 8.1 allow us to state and prove finitely additive counterparts of both the Borel-Cantelli lemmas and the Strong Law of Large Numbers (Theorems 9.1, 9.2, and 9.3). Finally, if γ is countably additive, our extension of π is just the conventional countably additive product measure (see Section 10) and so our methods are consistent with the usual procedures.

2. Basic Framework.

Throughout, probabilities and probability measures are finitely additive unless explicitly stated otherwise. Let N be the positive integers and X an arbitrary non-empty set. Let $H = X^N = X \times X \times \ldots$ and give H the product topology determined by assigning X the discrete topology. Subsets of H which are simultaneously closed and open in this topology will be referred to as clopen. (Strictly speaking, any of the topology in what follows is logically dispensable, and perhaps even a little distracting. But it does offer the convenience of familiarity.)

The following theorem states that any probability on the clopen sets of H may be extended uniquely by conventional methods to an algebra containing all the open sets. We are grateful to Dubins and Savage for letting us study some of their unpublished work on a similar extension. In particular, the key idea in the proof is a reduction principle for open sets (see Lemma 3.1 below) which we learned from them.

Theorem.

Let μ be a finitely additive probability defined on the clopen subsets of H. Then there is a unique finitely additive probability λ such that

(i) the domain of λ is an algebra $\mathcal{G}$ of sets containing the open sets;

(ii) λ extends μ;

(iii) if O is open and $\delta > 0$, there is a clopen set K such that $K \subseteq O$ and $\lambda(K) > \lambda(O) - \delta$;

(iv) $A \in \mathcal{G}$ if and only if, for every positive ϵ, there are O open, C closed such that $C \subseteq A \subseteq O$ and $\lambda(O - C) < \epsilon$.

The proof is given in Section 3. The only preliminaries needed to read Section 3 are the definitions of stop rule, and incomplete stop rule, given below.

At this point it is reasonable to inquire, for example, whether or not the $\mathcal{G}$ of Theorem 1 contains the G_δ's (countable intersections of open sets) but we do not know the answer to this. The next three paragraphs introduce a class of probabilities, the "probabilities determined by strategies," for which we have shown it does. This class, first considered by Dubins and Savage, is essential to our proof, which typically involves working with all of its members simultaneously.

Let X^* be the set of all finite sequences of members of X, including the empty one. A strategy σ is a function which assigns to each $p \in X^*$ a probability measure $\sigma(p)$, defined on all subsets of X. The probability assigned by σ to the empty sequence will be denoted σ_0. Informally, a strategy generates a chance sequence $(x_1, x_2, x_3, \ldots)$ of members of X in the following manner: let x_1 be chosen at random according to σ_0, let x_2 be chosen according to $\sigma(x_1)$, x_3 to be chosen according to $\sigma(x_1, x_2)$, and so on. For the special case considered in the introduction, $X = N$ and the corresponding strategy is the (constant) function which assigns γ to all members of N^*.

Before stating the precise sense in which a strategy determines a probability on the clopen subsets of H, a few preliminaries will be required. Let $p, q \in X^*$ and $h \in H$. Then pq is the member of X^* whose terms consist of the terms of p followed by the terms of q, and ph is the member of H whose terms consist of the terms of p followed by the terms of h. If $K \subseteq H$, $Kp = \{h \in H | ph \in K\}$. If w is a function defined on H, wp is defined by $wp: h \to w(ph)$, $h \in H$. If p consists of a single term

$x \in X$, Kp will be written Kx and wp will be written wx. If σ is a strategy and $p \in X^*$, $\sigma[p]$ is the conditional strategy defined by $\sigma[p](q) = \sigma(pq)$, all $q \in X^*$. If $p = (x)$, $x \in X$, $\sigma[p]$ will be written $\sigma[x]$. If S is any set 1_S, the indicator of S, is the function which is 1 on S and 0 off S.

For the next few paragraphs only, let $\mathcal{S}$ be the set of all strategies and $\mathcal{C}$ be the set of all bounded functions on H to the real line which are continuous when the latter is endowed with the discrete topology. Then there exists a unique real-valued function E, defined on $\mathcal{S} \times \mathcal{C}$, such that for every $(\sigma, w) \in \mathcal{S} \times \mathcal{C}$,

$$(2.1) \quad \begin{aligned} &E(\sigma, c) = c, \quad \text{for every constant } c, \\ &E(\sigma, w) = \int E(\sigma[x], wx)d\sigma_0(x). \end{aligned}$$

Further, for each $\sigma \in \mathcal{S}$, the function $w \to E(\sigma, w)$, $w \in \mathcal{C}$, is a positive linear functional on $\mathcal{C}$. Then the probability determined by a strategy σ is defined to be the set function $K \to E(\sigma, 1_K)$, K clopen.

The proof of the claims of the preceding paragraph, given in Section 2.8 of [2], is a transfinite recursion argument which turns on the fact that $\mathcal{C}$ can be arranged in an ordinally indexed hierarchy in such a way that wx, for $x \in X$, is always "below" w in the hierarchy, for all nonconstant $w \in \mathcal{C}$. The idea of the inductive step, is that once E has been defined for all pairs of the form (σ, wx) it can be extended up the hierarchy to (σ, w) by using (2.1). A precise definition of this hierarchy will not be presented here. It appears in Section 2.7 of [2]. An understanding of the contents of this section of [2], especially the notion of the structure of a function in $\mathcal{C}$, will be required in order to follow some of the main arguments of this paper. In particular, "structure of a clopen set K" will be used here to refer to the structure of 1_K.

To digress briefly, the process of defining E amounts, loosely, to extending all strategies simultaneously to linear functionals on $\mathcal{C}$. However, it is not necessary, only convenient, to extend this far in order to define probabilities uniquely on the clopen subsets of H. Nor is it necessary to work with the set of all strategies. The class $\mathcal{S}$ may be replaced by any $\mathcal{R} \subseteq \mathcal{S}$ provided it has the property that $\sigma \in \mathcal{R}$ and $x \in X$ implies $\sigma[x] \in \mathcal{R}$. The existence of E then takes the following more modest form. There is a unique function $m_{\mathcal{R}}$ defined for all (σ, K) with $\sigma \in \mathcal{R}$, K clopen such that

$$m_{\mathcal{R}}(\sigma, \phi) = 0, \qquad m_{\mathcal{R}}(\sigma, H) = 1,$$

$$m_{\mathcal{R}}(\sigma, K) = \int m_{\mathcal{R}}(\sigma[x], K_x) d\sigma_0(x),$$

for all $\sigma \in \mathcal{R}$, K clopen. Further, for each fixed σ, the function $K \to m_{\mathcal{R}}(\sigma, K)$, K clopen, is a probability. The proof is essentially the same as that for E. For an example, let $X = N$, γ be a probability on N, and $\mathcal{R}$ have as its sole member the strategy which assigns γ to all members of N^*. Then $m_{\mathcal{R}}$ gives the extension π claimed to exist in the introduction. Finally, the $m_{\mathcal{R}}$ are all consistent with E: $m_{\mathcal{R}}(\sigma, K) = E(\sigma, 1_K)$ for $\sigma \in \mathcal{R}$, K clopen, and any $\mathcal{R}$ (such that $\sigma \in \mathcal{R}$, $x \in X$ implies $\sigma[x] \in \mathcal{R}$).

If σ is a strategy, it determines, as just indicated the probability μ: $K \to E(\sigma, 1_K)$, K clopen. For this μ, let $\mathcal{G}(\sigma)$ be the algebra determined by Theorem 1, and with some harmless ambiguity, let σ be the probability λ determined by the same theorem. This convention will be in force from now on.

A <u>stop rule</u> is a function $r: H \to N$ such that if h, h' belong to H and $h_i = h'_i$ $i = 1,\ldots, r(h)$, then $r(h) = r(h')$. A clopen set K

is said to be <u>determined by time</u> r, provided that $h \in K$, $h' \in H$ and $h_i = h'_i$, $i = 1,\ldots, r(h)$ implies $h' \in K$. A sequence of stop rules $r_1, r_2, \ldots$ is said to be <u>strictly increasing</u> provided $r_1(h) < r_2(h) < \ldots$, for all $h \in H$.

An <u>incomplete stop rule</u> is a function $t\colon H \to N \cup \{\infty\}$ such that if $t(h) < \infty$ and $h_i = h'_i$, $i = 1,\ldots, t(h)$, then $t(h) = t(h')$. If t is an incomplete stop rule, the set $[t < \infty]$ $(= \{h \in H \mid t(h) < \infty\})$ is open. Conversely, if O is open there is an incomplete stop rule t such that $O = [t < \infty]$. One such t, the <u>minimal incomplete stop rule associated with</u> O, is defined by taking $t(h)$ to be the least k (if any) such that if $h' \in H$ and $h'_i = h_i$, $i = 1,\ldots, k$, then $h' \in O$; if no such k exists, $t(h) = \infty$.

Finally, there is a basic integration formula which will often be called upon. To render it more readable, the value of E at (σ, w) will be denoted $\int w(h)d\sigma(h)$. Also, set $p_n(h) = (h_1,\ldots, h_n)$ for $h \in H$, $n \in N$; and, if s is a stop rule set $p_s(h) = p_n(h)$ where $n = s(h)$. Then if σ is a strategy

$$\sigma(K) = \int \sigma[p_s(h)]Kp_s(h)d\sigma(h) \tag{2.2}$$

for all clopen K. The special case of (2.2) obtained by taking $s \equiv 1$, after a standard change of variable, is just the condition (2.1). The formula (2.2) is proved from this special case by induction on the structure of p_s. A slightly more general version of (2.2) is formula 2.7.1 in [2].

3. A Proof of Theorem 2.1.

Before proceeding to the proof, the following lemma and corollaries are required. These are well known in some circles (e.g., logicians) but included here for completeness.

Lemma 1.

Let $O, \check{O}$ be open sets in H. Then there exist $P, \check{P}$ open such that

$$P \subseteq O, \ \check{P} \subseteq \check{O},$$
$$P \cup \check{P} = O \cup \check{O},$$

and $P, \check{P}$ are disjoint.

Proof:

Let $t, \check{t}$ be the minimal incomplete stop rules associated with $O, \check{O}$ respectively. Set

$$P = [t < \infty, \ t \leq \check{t}], \quad \check{P} = [\check{t} < \infty, \ t > \check{t}].$$

The claimed properties of $P, \check{P}$ are easily verified. □

Corollary 1.

Let $K \subset O \cup \check{O}$ where K is clopen and $O, \check{O}$ are open. Then there exists $L, \check{L}$ clopen such that

$$L \subseteq O, \quad \check{L} \subseteq \check{O}$$
$$L \cup \check{L} = K$$

and $L, \check{L}$ are disjoint.

Proof:

Using the $P, \check{P}$ of the preceding lemma, set $L = K \cap P$, $\check{L} = K \cap \check{P}$. □

Corollary 2.

If $C, \check{C}$ are closed and disjoint there exists K clopen with $K \supseteq C$ and K disjoint from $\check{C}$.

Proof:

In Lemma 1, let O, $\check{O}$ be the complements of C, $\check{C}$ respectively. Then take K to be $\check{P}$. $\square$

Proof of Theorem 1:

The uniqueness of λ is easily verified. The existence proceeds in two stages. First, set

$$\eta(O) = \sup\{\mu(K) \mid K \text{ clopen, } K \subseteq O\}$$

for each open set O. Plainly,

$$\eta(O) + \eta(\check{O}) \leq \eta(O \cup \check{O})$$

for O, $\check{O}$ open disjoint. For the other inequality, which does not require that O, $\check{O}$ be disjoint, let $K \subseteq O \cup \check{O}$. Then, using the L, $\check{L}$ of Corollary 1,

$$\mu(K) = \mu(L) + \mu(\check{L}) \leq \eta(O) + \eta(\check{O})$$

It follows that η is finitely additive and subadditive on the open subsets of H. This completes the first stage of the extension.

For the second stage, set

$$\eta^*(A) = \inf\{\eta(O) \mid O \text{ open, } O \supseteq A\}$$

for every $A \subseteq H$; and $\mathcal{G}$ to be the collection of all $A \subseteq H$ satisfying:

For every $\epsilon > 0$, there exist O open, C closed with $C \subseteq A \subseteq O$ and $\eta^*(O - C) < \epsilon$.

Then verify the following, where A, B are arbitrary subsets of H.

(a) If $A \subseteq B$, $\eta^*(A) \leq \eta^*(B)$.

(b) $\eta^*(A \cup B) \leq \eta^*(A) + \eta^*(B)$. This requires the subadditivity of η.

(c) $\mathcal{G}$ is an algebra of sets.

(d) If $A \subseteq B$, $\eta^*(B) - \eta^*(A) \leq \eta^*(B - A)$.

(e) If $A \in \mathfrak{G}$,

$$\eta^*(A) = \sup\{\eta^*(C) \mid C \text{ closed, } C \subseteq A\}.$$

This follows easily from the definition of $\mathfrak{G}$, (d), and (a).

(f) If C, D are closed, disjoint

$$\eta^*(C \cup D) = \eta^*(C) + \eta^*(D) .$$

Given $\varepsilon > 0$, there is an O open such that $O \supseteq C \cup D$ and $\eta(O) \leq \eta^*(C \cup D) + \varepsilon$. Using Corollary 2 (or even the normality of H) there are open disjoint Q, $\check{Q}$ such that $Q \supseteq C$, $\check{Q} \supseteq D$. Then

$$\eta(O) \geq \eta(O \cap (Q \cup \check{Q})) = \eta(O \cap Q) + \eta(O \cap \check{Q}).$$

It follows that $\eta^*(C \cup D) \geq \eta^*(C) + \eta^*(D)$.

(g) If A, B belong to $\mathfrak{G}$ and are disjoint,

$$\eta^*(A \cup B) = \eta^*(A) + \eta^*(B).$$

Use (b), (e), (f).

(h) The open sets belong to $\mathfrak{G}$. To see this, let $\varepsilon > 0$ and O be open. There is a K clopen such that $K \subseteq O$ and $\mu(K) > \eta(O) - \varepsilon$. Then verify that $\eta^*(O - K) < \varepsilon$.

The theorem now follows by taking λ to be the restriction of η^* to $\mathfrak{G}$.

Definition.

If λ is as in Theorem 1,

$$\lambda^*(A) = \inf\{\lambda(O) \mid O \text{ open, } O \supseteq A\}$$

$$\lambda_*(A) = \sup\{\lambda(C) \mid C \text{ closed, } C \subseteq A\}$$

for all $A \subseteq H$.

Corollary 1.

If λ, $\mathfrak{G}$ are as in Theorem 1, $\mathfrak{G}$ coincides with the collection of all $A \subseteq H$ such that $\lambda^*(A) = \lambda_*(A)$.

Finally, the fact that open sets can be approximated in measure from within by clopen sets may be recast in the following form, which we have often found useful. This result was already known to Dubins and Savage.

Corollary 2.

Let O be open and t any incomplete stop rule for which $O = [t < \infty]$. Then, if λ is as in Theorem 1,

$$\lambda(O) = \sup_s \lambda[t \leq s]$$

where the supremum is taken over all stop rules s.

Proof:

The set $[t \leq s]$ is clopen and a subset of O. Further, for any clopen $K \subseteq O$ there is a stop rule s such that $s(h) = t(h)$, for all $h \in K$. (See Theorem 2.11.1 in [2].) These two facts, along with Theorem 2.1 (iii), imply the equality. $\square$

4. A Fubini Theorem.

The basic result of this section is a straightforward extension of the formula (2.2) of Section 2.

Theorem 1.

Let σ be a strategy. For every $A \subseteq H$,

$$\sigma^*(A) = \int \sigma[x]^*(Ax)d\sigma_0(x), \text{ and}$$
$$\sigma_*(A) = \int \sigma[x]_*(Ax)d\sigma_0(x).$$

Proof:

The conclusion holds for clopen sets. It is next established for open sets.

If $K \subseteq O$, $Kx \subseteq Ox$; so that

$$\sigma(O) = \sup \sigma(K) = \sup \int \sigma[x](Kx)d\sigma_0(x) \leq \int \sigma[x](Ox)d\sigma_0(x),$$

where the sup is taken over all clopen sets $K \subseteq O$.

For the opposite inequality, let $\varepsilon > 0$. If O is open, Ox is open, and so, for each $x \in X$ there is a $K(x) \subseteq Ox$ with $K(x)$ clopen and $\sigma[x]K(x) \geq \sigma[x](Ox) - \varepsilon$. Define K by $Kx = K(x)$ for all x. Then K is clopen, $K \subseteq O$, and

$$\sigma(O) \geq \sigma(K) = \int \sigma[x](Kx)d\sigma_0(x) \geq \int \sigma[x](Ox)d\sigma_0(x) - \varepsilon.$$

The argument just given can be easily adapted now to give the first equation in Theorem 1. The second equation follows from the first together with the fact that $\sigma_*(A) = 1 - \sigma^*(A^c)$. □

Corollary 1.

Let σ be a strategy and s a stop rule. Then, for every $A \subseteq H$,

$$\sigma^*(A) = \int \sigma[p_s(h)]^*(Ap_s(h))d\sigma(h), \text{ and}$$
$$\sigma_*(A) = \int \sigma[p_s(h)]_*(Ap_s(h))d\sigma(h).$$

This corollary extends Theorem 1, and is proved from it by induction on the structure of p_s.

Notice that, if $A \in \mathcal{G}(\sigma)$, then, by Theorem 1, $\int \sigma[x]^*(Ax)d\sigma_0(x) = \int \sigma[x]_*(Ax)d\sigma_0(x)$. Nevertheless, it can happen that $Ax \notin \mathcal{G}(\sigma[x])$ for all x even when $A \in \mathcal{G}(\sigma)$.

5. The Measure of Countable Intersections.

It is easy to verify that

$$(5.1) \qquad P\left(\bigcap_{1}^{\infty} A_n\right) = P(A_1) \prod_{2}^{\infty} P(A_n | A_1 \cap \dots \cap A_{n-1})$$

when P is a countably additive probability and both sides are well-defined. For a finitely additive P, the left side of (5.1) can be smaller than the right. However, the theorems of this section give formulas analogous to (5.1) which are appropriate even in a finitely additive setting. In particular, Theorem 4 states that (5.1) holds for "independent events."

Let $\{K_n\}$ be a sequence of clopen sets and let $\{r_n\}$ be a strictly increasing sequence of stop rules such that, for every positive integer n, K_n is determined by time r_n. Define, for every $n \in N$ and $h \in H$, $q_n(h) = p_{r_n}(h)$. Finally, let $\{\alpha_n\}$ be a sequence of numbers satisfying $0 \leq \alpha_n \leq 1$ for all n and let σ be a strategy.

Theorem 1.

If $\sigma(K_1) \geq \alpha_1$ and if, for all $n = 1,2,\dots$ and all $h \in \bigcap_1^n K_i$, $\sigma[q_n(h)](K_{n+1}q_n(h)) \geq \alpha_{n+1}$, then $\sigma(\bigcap_1^\infty K_i) \geq \prod_1^\infty \alpha_i$.

Proof:

The set $\bigcap_1^\infty K_i$ is closed. Let K be clopen and $K \supseteq \bigcap_1^\infty K_i$. By Theorem 2.1, it suffices to show $\sigma(K) \geq \prod_1^\infty \alpha_i$.

The argument is by induction on the structure of K. We can and do assume $\alpha_i > 0$ for all i.

Suppose K has structure 0. Then either $K = H$ or $K = \emptyset$. If $K = H$, then $\sigma(K) = 1 \geq \prod_1^\infty \alpha_i$. We show K cannot be empty by constructing a history $h \in \bigcap_1^\infty K_i$. Since $\sigma(K_1) \geq \alpha_1 > 0$, there exists $h^1 \in K_1$. Since $\sigma[q_1(h^1)](K_2 q_2(h^1)) \geq \alpha_2 > 0$, there exists $h^2 \in K_2$ such that h^2 agrees with h^1 up to time $r_1(h^1)$. Continue in this fashion to define $h^n \in K_n$ such that h^n agrees with h^{n-1} up to time $r_{n-1}(h^{n-1})$. Then let h be that history which agrees with h^n up to time $r_n(h^n)$ for all n. Since K_n is determined by time r_n and $h^n \in K_n$, we have $h \in K_n$ for all n.

For the inductive step, assume the desired result for sets of structure less than α and suppose K has structure $\alpha > 0$. Then, for all h, $Kq_1(h) \supseteq (\bigcap_1^\infty K_n)q_1(h) = \bigcap_1^\infty (K_n q_1(h))$ and $Kq_1(h)$ has structure less than α.

Fix $h \in K_1$. Set $q = q_1(h)$ and define $\sigma' = \sigma[q]$, $K_n' = K_{n+1}q$, and $r_n'(h') = r_{n+1}(qh') - r_1(h)$ for $h' \in H$. Then $\sigma'(K_1') = \sigma[q_1(h)](K_2 q_1(h)) \geq \alpha_2$. Also, if $h' \in \bigcap_1^n K_i'$, then $qh' \in \bigcap_1^{n+1} K_i$ and

$h' \in \bigcap_{i=1}^n K_i'$ $\sigma[q_n'(h')](K$ > 0

$$\sigma'[q_n'(h')](K_{n+1}' q_n'(h')) = \sigma[q_{n+1}(qh')](K_{n+2} q_{n+1}(qh')) \geq \alpha_{n+2}.$$

By the inductive assumption, if $h \in K_1$, then $\sigma[q_1(h)](Kq_1(h)) \geq \prod_2^\infty \alpha_n$. Hence, by equation (2.2),

$$\sigma(K) = \int \sigma[q_1(h)](Kq_1(h))d\sigma(h) \geq \int_{K_1} \sigma[q_1(h)](Kq_1(h))d\sigma(h)$$

$$\geq \sigma(K_1) \prod_2^\infty \alpha_n \geq \prod_1^\infty \alpha_n. \ \square$$

Let $B_n = \{h \in \bigcap_1^n K_i : \sigma[q_n(h)](K_{n+1}q_n(h)) \geq \alpha_n\}$ for $n = 1,2,\ldots.$

It would be convenient to replace the assumption in Theorem 1 that $B_n = \bigcap_1^n K_i$ by the milder one that $\bigcap_1^n K_i - B_n$ has small probability. An example shows there is no hope for such a generalization.

<u>Example.</u>

Let $X = N$; let σ_0 be a finitely additive probability on all subsets of N such that $\sigma_0(\{n\}) = 0$ for all n; let $\sigma[n]$ assign mass one to the history $(n, n,\ldots)$ for all n; let $r_n = n$, $K_n = \{h: h_n \geq 1\}$, and $\alpha_n = 1$. Then $\sigma(B_n) = 1$ for all n, but $\bigcap K_n = \phi$.

The next result is a simple inequality which goes in the opposite direction from that of Theorem 1.

Theorem 2.

Assume $\sigma(K_1) \leq \alpha_1$ and let $C_n = \{h \in \bigcap_1^n K_i : \sigma[q_n(h)](K_{n+1}q_n(h)) > \alpha_{n+1}\}$ for $n = 1,2,\ldots.$ Then

$$\sigma\left(\bigcap_1^n K_i\right) \leq \prod_1^n \alpha_i + \sum_1^{n-1} \sigma(C_i), \text{ for } n = 2,3,\ldots, \tag{5.3}$$

and, hence,

$$\sigma\left(\bigcap_1^\infty K_i\right) \leq \prod_1^\infty \alpha_i + \sum_1^\infty \sigma(C_i).$$

Proof:

Assume (5.3) is true for n. Then

$$\begin{aligned}
\sigma\left(\bigcap_1^{n+1} K_i\right) &= \int_{\bigcap_1^n K_i} \sigma[q_n(h)](K_{n+1}q_n(h))d\sigma(h) \\
&= \int_{C_n} \sigma[q_n(h)](K_{n+1}q_n(h))d\sigma(h) + \int_{\bigcap_1^n K_i - C_n} \sigma[q_n(h)](K_{n+1}q_n(h))d\sigma(h) \\
&\leq \sigma(C_n) + \alpha_{n+1}\left(\prod_1^n \alpha_i + \sum_1^{n-1}\sigma(C_i)\right) \\
&\leq \prod_1^{n+1} \alpha_i + \sum_1^n \sigma(C_i). \ \square
\end{aligned}$$

The next result is immediate from Theorems 1 and 2.

Theorem 3.

If $\sigma(K_1) = \alpha_1$ and if, for all $n = 1,2,\ldots$ and all $h \in \bigcap_1^n K_i$, $\sigma[q_n(h)](K_{n+1}q_n(h)) = \alpha_{n+1}$, then $\sigma\left(\bigcap_1^\infty K_i\right) = \prod_1^\infty \alpha_i$.

Now let $\{\gamma_n\}$ be a sequence of probabilities defined on all subsets of X. Define the strategy $\sigma = \gamma_1 \times \gamma_2 \times \ldots$ by $\sigma_0 = \gamma_1$ and, for all p of length n, $\sigma(p) = \gamma_{n+1}$. Such a strategy is said to be independent. Notice that $\sigma[p] = \gamma_{n+1} \times \gamma_{n+2} \times \ldots$ for every p of length n.

Theorem 4.

Let $\sigma = \gamma_1 \times \gamma_2 \times \ldots$ be an independent strategy and let $A_n \subseteq X$ for $n = 1,2,\ldots.$ Then

$$\sigma(A_1 \times A_2 \times \dots) = \gamma_1(A_1)\ \gamma_2(A_2)\dots.$$

Proof:

Let $K_n = \{h: h_n \in A_n\}$, and let $r_n = n$. Then $\bigcap_1^\infty K_n = A_1 \times A_2 \times \dots$ and $\sigma[q_n(h)](K_{n+1}q_n(h)) = \gamma_{n+1}(A_{n+1})$ for all n and h. □

Further results on independent strategies are in Section 9.

Note:

Some readers may prefer to skip the next three sections and proceed directly to Section 9 which contains the Borel-Cantelli lemmas and the strong law. Although Theorem 9.2 depends on the intervening sections, the strong law does not.

6. A "Fatou Equation."

Recall that, if $\{Z_n\}$ is a uniformly bounded sequence of random variables on a (countably additive) probability space, then, by Fatou's Lemma,

$$E(\limsup_{n\to\infty} Z_n) \geq \limsup_{n\to\infty} E(Z_n). \tag{6.1}$$

If the lim sup on the right is taken over the directed set of stopping variables s, then, as shown in [3], an equality results:

$$E(\limsup_{n\to\infty} Z_n) = \limsup_{s\to\infty} E(Z_s). \tag{6.2}$$

Also, (6.1) follows easily from (6.2) in the countably additive case. For finitely additive probabilities (6.1) is false, but, suitably restated, (6.2) continues to hold. Rather than give that result here, we prove a special case from which it may be derived rather easily using the methods of Section 3 of [4].

First, some notation. Let $A \subseteq X$, and let σ be a strategy. Set

$$A(\sigma) = \limsup_{s\to\infty} \sigma\{h|h_{s(h)} \in A\}.$$

The lim sup is over all stop rules s and, by definition, means $\inf_r \sup_{s\geq r} \sigma\{h|h_{s(h)} \in A\}$.

Also, for $n \in N$, define the function x_n by $x_n: h \to h_n$, $h \in H$; and, if s is a stop rule, let $x_s: h \to h_n$ where $n = s(h)$, $h \in H$. Then, for example, $[x_n \in A]$ denotes $\{h: h_n \in A\}$.

In the statement of the next theorem, the counterpart of Z_n in (6.2) is the indicator function of the set $[x_n \in A]$.

Theorem 1.

Let σ be a strategy and $A \subseteq X$. Then the set $[x_n \in A \text{ i.o.}]$ is in $\mathcal{G}(\sigma)$ and

(6.3) $\sigma[x_n \in A \text{ i.o.}] = A(\sigma)$.

(The symbol "i.o." is short for "infinitely often.")

Proof:

Let $B = [x_n \in A \text{ i.o.}]$. Also, for each stop rule s, let $\hat{s}$ be the incomplete stop rule defined by

$$\hat{s}(h) = \text{least } n \geq s(h), \text{ if any, such that } h_n \in A,$$
$$= \infty \text{ if } h_n \notin A \text{ for all } n \geq s(h).$$

By Corollary 3.1, it suffices to show $\sigma^*(B) \leq A(\sigma) \leq \sigma_*(B)$.

First we show

(6.4) $\sigma^*(B) \leq A(\sigma)$.

Let $\epsilon > 0$. Choose a stop rule s such that, if $r \geq s$, then $\sigma[x_r \in A] < A(\sigma) + \epsilon$. Notice $B \subseteq [\hat{s} < \infty]$. Let r be a stop rule and $r \geq s$. Set $r' = r \wedge \hat{s}$. Then $[\hat{s} \leq r] = [\hat{s} = r'] \subseteq [x_{r'} \in A]$. Hence

$$\sigma^*(B) \leq \sigma[\hat{s} < \infty] = \sup_{r \geq s} \sigma[\hat{s} \leq r] \quad \text{(Corollary 3.2)}$$
$$\leq \sup_{r \geq s} \sigma[x_{r'} \in A] \leq A(\sigma) + \epsilon.$$

The last inequality holds since $(r \geq s \Rightarrow r' \geq s)$. This proves (6.4).

It remains to show

(6.5) $\sigma_*(B) \geq A(\sigma)$.

The idea of the proof is to construct an increasing sequence of stop rules with the property that one is likely to visit A at least once between each pair of adjacent stop rules. Then an application of Theorem 5.1 will give a lower bound to the probability of visiting A at least once between every adjacent pair and thus to the probability of visiting A infinitely often.

We need two lemmas.

Lemma 1.

For every pair of positive numbers ϵ and δ, every strategy σ, and every stop rule r', there is a stop rule $r \geq r'$ such that

$$\sigma\{h: A(\sigma[p_r(h)]) > 1 - \epsilon\} > A(\sigma) - \delta.$$

Proof of Lemma 1.

By Lemma 3.7.1 of [2], given ϵ' and δ' positive, there exists $r \geq r'$ such that $\sigma\{h: A(\sigma[p_r(h)]) \geq \epsilon'\} \leq A(\sigma) + \delta'$. Let $g(h) = A(\sigma[p_r(h)])$ for $h \in H$ and let $\beta = \sigma[g > 1 - \epsilon]$. It suffices to show that, for some choice of ϵ' and δ', $\beta > A(\sigma) - \delta$.

Notice $0 \leq g \leq 1$ and, by formula 3.3.1 of [2], $A(\sigma) = \int g d\sigma$. Thus

$$A(\sigma) = \int_{[g>1-\epsilon]} g d\sigma + \int_{[\epsilon' \leq g \leq 1-\epsilon]} g d\sigma + \int_{[g<\epsilon']} g d\sigma$$

$$\leq \beta + (A(\sigma) + \delta' - \beta)(1-\epsilon) + \epsilon', \text{ whenever } \epsilon' < 1 - \epsilon.$$

Hence,

$$\beta \geq A(\sigma) - \epsilon^{-1}(\delta'(1-\epsilon) + \epsilon')$$

and the desired inequality holds for sufficiently small ϵ' and δ'. □

Let v be the incomplete stop rule given by

$$v(h) = \text{least } n, \text{ if any, such that } h_n \in A,$$
$$= +\infty \text{ if } h_n \notin A \text{ for all } n.$$

Lemma 2.

For every pair of positive numbers ϵ and δ, and every strategy σ, there is a stop rule r such that

$$\sigma\{h: v(h) \leq r(h) \text{ and } A(\sigma[p_r(h)]) > 1 - \epsilon\} > A(\sigma) - \delta.$$

Proof of Lemma 2.

Let ϵ_1, ϵ_2, ϵ_3, and ϵ_4 be small positive numbers to be specified later.

Notice that $A(\sigma) = \inf_s \sup_{r\geq s} \sigma[x_r \in A] = \inf_s \sigma[\hat{s} < \infty]$. For the second equation, use Corollary 3.2. Thus, we can and do choose a stop rule r_0 so that

$$(6.6) \qquad \sigma[\hat{r}_0 < \infty] < A(\sigma) + \epsilon_1.$$

Now choose a stop rule r_1 so that

$$(6.7) \qquad r_1 \geq r_0 \text{ and } \sigma[x_{r_1} \in A] > A(\sigma) - \epsilon_2.$$

Finally, using Lemma 1, choose a stop rule r satisfying

$$(6.8) \qquad r \geq r_1 \text{ and } \sigma[A(\sigma[p_r]) > 1 - \epsilon_3] > A(\sigma) - \epsilon_4.$$

The stop rule r works if the ϵ_i are sufficiently small.

To see this, let $E = [A(\sigma[p_r]) > 1 - \epsilon_3]$ and $F = [v \leq r]$. Assume $\epsilon_3 < \epsilon$. Then we need $\sigma(E \cap F) > A(\sigma) - \delta$. By (6.8), $\sigma(E) > A(\sigma) - \epsilon_4$. Assume $\epsilon_4 < \delta/2$. Then it suffices to get $\sigma(E - F) \leq \delta/2$.

Notice that $F \supseteq [\hat{r}_0 \leq r] \supseteq [\hat{r}_0 \leq r_1]$. Set $G = [\hat{r}_0 \leq r_1]$. Then $E\text{-}F \subseteq E\text{-}G$ and, hence, it suffices to get $\sigma(E\text{-}G) \leq \delta/2$.

For each $h \in E$, choose a stop rule $s(h)$ such that $\sigma[p_r][x_{s(h)} \in A] > 1-\epsilon_3$. for each $h \in E^c$, let $s(h)$ be an arbitrary stop rule. Let r_2 be the composition of r with the family $s(\cdot)$; i.e., let

$$r_2(h) = r(h) + s(h)(h_{r(h)+1}, h_{r(h)+2}, \ldots).$$

Then

$$\sigma(E \cap [x_{r_2} \in A]) = \int_E \sigma[p_r(h)][x_{s(h)} \in A]d\sigma(h) \geq \sigma(E)(1-\epsilon_3) \geq \sigma(E) - \epsilon_3.$$

Hence,

$$(6.9) \qquad \sigma(E \cap [x_{r_2} \in A]^c) \leq \epsilon_3.$$

Also, we have

$$\begin{aligned}(6.10) \qquad \sigma(G^c \cap [x_{r_2} \in A]) &= \sigma([\hat{r}_0 > r_1] \cap [x_{r_2} \in A]) \leq \sigma[r_1 < \hat{r}_0 \leq r_2] \\ &= \sigma[\hat{r}_0 \leq r_2] - \sigma[\hat{r}_0 \leq r_1] \leq \sigma[\hat{r}_0 < \infty] - \sigma[x_{r_1} \in A] \\ &\leq \epsilon_1 + \epsilon_2.\end{aligned}$$

The last inequality follows from (6.6) and (6.7).

By (6.9) and (6.10),

$$\sigma(E - G) = \sigma(E \cap G^c \cap [x_{r_2} \in A]) + \sigma(E \cap G^c \cap [x_{r_2} \in A]^c) \leq \varepsilon_1 + \varepsilon_2 + \varepsilon_3.$$

Choose the ε_i so that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 < \delta/2$ and the proof is complete. □

Now we are ready to prove (6.5).

Let $\varepsilon_0, \varepsilon_1, \ldots$ be small positive numbers. Choose a stop rule r_1 by Lemma 2 so that

$$\sigma[v \leq r_1, A(\sigma[q_1]) > 1 - \varepsilon_1] > A(\sigma) - \varepsilon_0,$$

where $q_1(h) = p_{r_1}(h)$. Define

$$K_1 = [v \leq r_1, A(\sigma[q_1]) > 1 - \varepsilon_1].$$

Suppose now that the stop rules $r_1, \ldots, r_n$ and clopen sets $K_1, \ldots, K_n$ are defined. Let $q_i(h) = p_{r_i}(h)$ and assume:

(a) $r_1(h) < \ldots < r_n(h)$ for all h,

(b) $K_i = [r_{i-1} \leq r_i, A(\sigma[q_i]) > 1 - \varepsilon_i]$ for $i = 2, \ldots, n$,

(c) $\sigma[q_{i-1}(h)](K_i q_{i-1}(h)) \geq 1 - \varepsilon_{i-1}$ for $h \in K_{i-1}$, $i = 2, \ldots, n$.

We want to define r_{n+1} and K_{n+1} so that (a), (b), and (c) continue to hold.

Let $h \in K_n$. By (b), $A(\sigma[q_n(h)]) > 1 - \varepsilon_n$. So, by Lemma 2, there is a stop rule $s(h)$ such that

$$\sigma[q_n(h)]\{h': v(h') \leq s(h)(h'), A(\sigma[q_n(h)][p_{s(h)}(h')]) > 1 - \varepsilon_{n+1}\} > 1 - \varepsilon_n. \tag{6.11}$$

For $h \in K_n^c$, let $s(h)$ be an arbitrary stop rule. Let r_{n+1} be the composition of r_n with $s(\cdot)$; i.e., let

$$r_{n+1}(h) = r_n(h) + s(h)(h_{r_n(h)+1}, h_{r_n(h)+2}, \ldots).$$

Define K_{n+1} by (b) with $i = n + 1$. Then (c) also holds with $i = n + 1$ and, in fact, is just (6.11).

By induction, we have (a), (b), and (c) for all n. By Theorem 5.1,

$$\sigma\left(\bigcap_1^\infty K_n\right) \geq (A(\sigma) - \varepsilon_0) \prod_1^\infty (1 - \varepsilon_n).$$

Also, if $h \in \bigcap_1^\infty K_n$, then, for all n, there is an i such that $r_n(h) \leq i \leq r_{n+1}(h)$ and $h_i \in A$. Hence, $\bigcap_1^\infty K_n \subseteq B$. Since the ε_n are arbitrary, equation (6.5) follows.

The proof of Theorem 1 is now complete.

7. A Generalization of the "Fatou Equation."

The main object of this section is to prove the following generalization of Theorem 6.1.

Theorem 1.

Let σ be a strategy on H and let $K_1, K_2, \ldots$ be a sequence of clopen subsets of H. Then the set $B = \{h: h \in K_n \text{ i.o.}\}$ is in $\mathcal{G}(\sigma)$. Further suppose that $r_1, r_2, \ldots$ is a strictly increasing sequence of stop rules on H such that, for every n, K_n is determined by time r_n. If s is a stop rule on H and $h \in H$, let $\check{s}(h) = \min\{n: r_n(h) \geq s(h)\}$. Then $\sigma(B) = \limsup_{s \to \infty} \sigma\{h: h \in K_{\check{s}(h)}\}$. (The lim sup is taken over all stop rules s on H.)

It should be remarked that, for a stop rule s, the function $\check{s}$ defined above is not necessarily a stop rule. However, the set $\{h: h \in K_{\check{s}(h)}\}$ is clopen and, in fact, is determined by time r_s, where $r_s(h) = r_{s(h)}(h)$.

The idea of the proof is familiar enough to probabilists. It is that of replacing the set of states of a discrete stochastic process by a new state space consisting of all finite sequences of members of the original state space. However, in our context, there is an obstacle to moving back and forth freely between the two situations. In transferring probabilities from the old situation to the new one, the property of being determined by a strategy must be preserved, because our methods only apply to probabilities which have this property. The first three lemmas are part of a rather indirect way of overcoming this obstacle. A more explicit approach is possible, but the notation involved , at least for us, was quite unappealing.

For the first lemma, let Y be a non-empty set and $G = Y^N$. Consider G to have the product topology, each Y having the discrete topology.

Lemma 1.

Let $\varphi: H \to G$ in such a way that $C = \varphi(H)$ is closed in G and φ is a homeomorphism of H to C (relative topology). Let σ be a strategy on H and λ be the finitely additive probability defined by

$$\lambda(L) = \sigma(\varphi^{-1}(L)), \quad L \text{ clopen in } G.$$

Then

$$\lambda^*(A) = \sigma^*(\varphi^{-1}(A))$$

$$\lambda_*(A) = \sigma_*(\varphi^{-1}(A))$$

for all $A \subseteq H$.

<u>Proof</u>:

Using λ to denote the extension to $\mathcal{G}(\lambda)$ as in Section 3, the first step is to show

$$\lambda(Q) = \sigma(\varphi^{-1}(Q))$$

for each Q open in G. The inequality $\lambda(Q) \leq \sigma(\varphi^{-1}(Q))$ is trivial. In the other direction, let $K \subseteq \varphi^{-1}(Q)$. Then $\varphi(K) \subseteq Q$ and $\varphi(K)$ is closed in G, the latter by the assumptions on φ. By Corollary 3.2 there is an L, clopen in G, such that

$$\varphi(K) \subseteq L \subseteq Q.$$

Then, since $K \subseteq \varphi^{-1}(L)$,

$$\sigma(K) \leq \sigma(\varphi^{-1}(L)) = \lambda(L) \leq \lambda(Q).$$

Thus $\lambda(Q) = \sigma(\varphi^{-1}(Q))$. The remainder of the proof is routine. □

<u>Lemma 2</u>.

Let Λ be a collection of finitely additive probabilities on the clopen subsets of H. Suppose that Λ is closed in the following sense: If $\lambda \in \Lambda$, there exists λ^0, $(\lambda^x; x \in X)$, where λ^0 is a probability on X and $(\lambda^x; x \in X)$ is a family of members of Λ, such that

$$\lambda(K) = \int \lambda^x(Kx)d\lambda^0(x)$$

for all K clopen in H.

Then, for each $\lambda \in \Lambda$, there is a strategy σ on H such that $\sigma(K) = \lambda(K)$ for all K clopen in H.

Proof:

The following inductive definition helps in avoiding many parentheses: Let $\lambda \in \Lambda$. If p is empty, set

$$\lambda^p = \lambda .$$

Suppose λ^p has been defined as a member of Λ for all p of length m, where m is a fixed non-negative integer. Then, if q has length $m + 1$, it is uniquely of the form px, whose p has length m and $x \in X$. Set

$$\lambda^q = (\lambda^p)^x .$$

This definition implies

$$(*) \qquad (\lambda^z)^p = (\lambda)^{zp}$$

for all $\lambda \in \Lambda$, $z \in X$, and p a finite sequence of elements of X.

The next step is to associate a strategy $\rho\lambda$ with each $\lambda \in \Lambda$ in the following manner:

$$(\rho\lambda)(p) = (\lambda^p)^0 .$$

For each x, the conditional strategy $(\rho\lambda)[x]$ satisfies

$$(\rho\lambda)[x] = \rho\lambda^x,$$

as is seen by applying (*). A proof by induction on the structure of K (clopen) then shows that

$$(\rho\lambda)(K) = \lambda(K)$$

for all $\lambda \in \Lambda$. □

For the remainder of this section, let Y be the set of all finite sequences (including the empty one) of elements of X, and let $G = Y^N$. To every $p \in Y$ and every sequence $r_1, r_2, \ldots$ of pointwise strictly increasing stop rules on H, associate a map $\varphi: H \to G$ defined by

(7.1) $\varphi(h) = (pq_1(h), pq_2(h), \ldots),$

where $q_n(h) = p_{r_n}(h)$. Let Φ be the collection of all such φ. It is straightforward to check that each $\varphi \in \Phi$ satisfies the assumptions of Lemma 1.

Lemma 3.

Let σ be a strategy on H and $\varphi \in \Phi$. Then there is a strategy $\hat{\sigma}$ on G such that $\hat{\sigma}(L) = \sigma(\varphi^{-1}(L))$ for all clopen L contained in G.

Proof:

For each strategy σ on H and each $\varphi \in \Phi$, let $\lambda = \varphi\sigma$ denote the finitely additive probability on the clopen sets L in G which satisfies

$$\lambda(L) = (\varphi\sigma)(L) = \sigma(\varphi^{-1}(L)).$$

Denote by Λ the collection of all λ obtained in this fashion. It suffices to show that Λ satisfies the assumptions of Lemma 2 if H is replaced by G there.

Let $\lambda = \varphi\sigma \in \Lambda$ and assume φ is given by Equation (7.1). Set $\eta(h) = pq_1(h)$ and define λ^0 to be the distribution of $\eta(h)$ under σ; that is, for $B \subseteq Y$, $\lambda^0(B) = \sigma\{h: \eta(h) \in B\}$. Let $y \in Y$. If $y = \eta(h)$ for some h, let φ^y be that element of Φ which is associated with y and the sequence of stop rules $\{r_n[q_1(h)]\}_{n\geq 2}$, and let $\sigma^y = \sigma[q_1(h)]$. Notice that these definitions are unambiguous. If y is not of the form $\eta(h)$ for some h (an event of λ^0 probability zero), let φ^y be an arbitrary element of Φ and let σ^y be an arbitrary strategy on H. Set $\lambda^y = \varphi^y\sigma^y$.

Let $C = \varphi(H)$ and $C^y = \varphi^y(H)$ for all y. Notice that $C^y = Cy$ for $y = \eta(h)$. Another elementary formula which is trying to check is

(7.2) $\varphi(K)y = \varphi^y(Kq_1(h))$

which holds for $K \subseteq H$ and $y = \eta(h)$.

Now suppose L is a clopen subset of G. Let $K = \varphi^{-1}(L) = \varphi^{-1}(L \cap C)$. Since φ is a homeomorphism from H to C, K is clopen in H. Now compute

$$
\begin{aligned}
\lambda(L) &= \sigma(K) \\
&= \int \sigma^{\eta(h)}(Kq_1(h))d\sigma(h) \quad \text{(by formula 2.2 of Section 2)} \\
&= \int \lambda^{\eta(h)}(\varphi^{\eta(h)}(Kq_1(h)))d\sigma(h) \\
&= \int \lambda^{\eta(h)}(\varphi(K)\eta(h))d\sigma(h) \quad \text{(by (7.2))} \\
&= \int \lambda^{y}(\varphi(K)y)d\lambda^{0}(y) \\
&= \int \lambda^{y}(Ly \cap Cy)d\lambda^{0}(y) \quad (\text{since } \varphi(K) = L \cap C) \\
&= \int \lambda^{y}(Ly)d\lambda^{0}(y) \quad (\text{since } \lambda^{y}(Cy) = \lambda^{y}(C^{y}) = 1.
\end{aligned}
$$

Thus, by the previous lemma, each λ arises from a strategy. □

Now we are ready to prove the main theorem.

<u>Proof of Theorem 1.</u>

For each n, let r_n be a stop rule on H such that K_n is determined by time r_n. Assume without loss of generality that the r_n are strictly increasing. (If they were not strictly increasing, each r_n could be replaced by s_n where $s_1 = r_1$ and $s_{n+1} = \max\{r_{n+1}, s_n + 1\}$.) Let $q_n(h) = p_{r_n}(h)$ for all n and h, and let $\varphi: H \to G$ be defined by (7.1) with p the empty sequence. Set

$$A = \{y \in Y: \text{for some } n \in N \text{ and } h \in H, y = q_n(h) \text{ and } h \in K_n\}.$$

Then

$$\varphi(B) = \{g \in G: g_n \in A \text{ i.o.}\} \cap C,$$

where $C = \varphi(H)$. A fact useful for checking this equation is that, given $n, m \in N$ and $h, h' \in H$, if $q_n(h) = q_m(h')$, then $n = m$.

Let $\lambda(L) = \sigma(\varphi^{-1}(L))$ for L a clopen subset of G. By Lemma 3, λ arises from a strategy. It follows from Theorem 6.2 that $\varphi(B) \in \mathcal{G}(\lambda)$ and then, from Lemma 1, that $B \in \mathcal{G}(\sigma)$.

For $n \in N$, let y_n be defined by $y_n: g \to g_n$, $g \in G$; and, if r is a stop rule on G, $y_r: g \to g_n$ where $n = r(g)$, $g \in G$. Then from Theorem 6.1, it follows that $\sigma(B) = \lambda(\varphi(B)) = \lambda[y_n \in A \text{ i.o.}] = \limsup_{r\to\infty} \lambda[y_r \in A]$. What remains to be shown is that

$$\limsup_{r\to\infty} \lambda[y_r \in A] = \limsup_{s\to\infty} \sigma\{h: h \in K_{\check{s}(h)}\}, \tag{7.3}$$

where the first lim sup is over stop rules r on G and the second over stop rules s on H.

To each stop rule r on G associate a stop rule $s = \alpha(r)$ on H where, for $h \in H$, $s(h) = r_n(h)$ if $r(\varphi(h)) = n$. It's easy to check that s is a stop rule and then that the following hold:

(a) $\check{s}(h) = r(\varphi(h))$ for all $h \in H$,

(b) $\varphi^{-1}[y_r \in A] = \{h: h \in K_{\check{s}(h)}\}$,

(c) $\lambda[y_r \in A] = \sigma\{h: h \in K_{\check{s}(h)}\}$.

Now let s be a stop rule on H. Define a stop rule $r = \beta(s)$ on G as follows: If $g = \varphi(h)$ for some h, let

$$r(g) = \min\{n: r_n(h) \geq s(h)\}.$$

Suppose $g \notin \varphi(H)$. If there is a $g' \in \varphi(H)$ such that g and g' agree in the first $r(g')$ coordinates, let $r(g) = r(g')$. If there is no such g', let $r(g)$ be the first n such that no history in $\varphi(H)$ agrees with g up to time n. Such an n must exist since $\varphi(H)$ is closed. (This construction is essentially the same as that of Theorem 2.11.1 in [2].) It is straightforward to check that r is a stop rule and that equations (a), (b), and (c) continue to hold.

To prove (7.3) let r_0 be a stop rule on G and set $s_0 = \alpha(r_0)$. Let s be a stop rule on H with $s \geq s_0$. Let $r = \beta(s_0)$ and $r' = r \vee r_0$.

Then $r' \geq r_0$ and r' agrees with r on $\varphi(H)$. Hence, $\lambda[y_{r'} \in A] = \lambda[y_r \in A]$ $= \sigma\{h: h \in K_{\check{s}(h)}\}$ and $\sup_{r \geq r_0} \lambda[y_r \in A] \geq \sup_{s \geq s_0} \sigma\{h: h \in K_{\check{s}(h)}\}$. It follows now that

$$\limsup_{r \to \infty} \lambda[y_r \in A] \geq \limsup_{s \to \infty} \sigma\{h: h \in K_{\check{s}(h)}\} .$$

The opposite inequality is proved similarly. □

If $\{E_n\}$ is a sequence of events in a (countably additive) probability space, then $P[E_n \text{ i.o.}] = \lim_{n \to \infty} P(\bigcup_{i \geq n} E_i)$. The next theorem gives an analogous result for strategies.

Theorem 2.

Let σ be a strategy and $\{K_n\}$ a sequence of clopen sets. For each stop rule s, let $A_s = \{h: h \in \bigcup_{n \geq s(h)} K_n\}$. Then the A_s are open and $\sigma(A_s)$ converges monotonically down to $\sigma[K_n \text{ i.o.}]$ as $s \to \infty$ through the directed set of stop rules.

Proof:

Let s be a stop rule and $h \in A_s$. Set $n = s(h)$ and $p = p_n(h)$. Then $A_s p = \bigcup_{i \geq n} K_i p$ is open. Furthermore the set of histories h whose first n coordinates agree with p and whose coordinates from $n + 1$ on form an element of $A_s p$ is an open set containing h and contained in A_s. Therefore A_s is open.

Notice also that $A_s \supseteq A_r$ when $s \leq r$ so clearly $\lim_{s \to \infty} \sigma(A_s)$ exists and equals $\inf_s \sigma(A_s)$.

Now, as in Theorem 1, let $\{r_n\}$ be a strictly increasing sequence of stop rules such that, for every n, K_n is determined by time r_n. For each stop rule s and $h \in H$, let $q_s(h) = p_{r_s}(h)$ where $r_s(h) = r_{s(h)}(h)$. It's easy to verify that r_s is a stop rule and so, by Corollary 4.1,

$$\sigma[K_n \text{ i.o.}] = \int \sigma[q_s(h)]([K_n \text{ i.o.}]q_s(h))d\sigma(h)$$
$$\leq \int \sigma[q_s(h)](\bigcup_{n\geq s(h)} K_n q_s(h))d\sigma(h)$$
$$= \sigma(A_s).$$

Take the infimum over s to get $\sigma[K_n \text{ i.o.}] \leq \lim_{s\to\infty} \sigma(A_s)$.

The opposite inequality uses the formula of the previous theorem which can be written as

$$\sigma[K_n \text{ i.o.}] = \inf_r \sup_{s\geq r} \sigma\{h: h \in K_{\check{s}(h)}\}.$$

For every stop rule r and $h \in H$, let $t_r(h)$ be the least $r_n(h)$ (if any) such that $r_n(h) \geq r(h)$ and $h \in K_n$, and let $t_r(h) = \infty$ if there is no such $r_n(h)$. Then t_r is an incomplete stop rule and, by Corollary 3.2 and the fact that $r_s \geq s$ for every stop rule s,

$$\sigma[t_r < \infty] = \sup_s \sigma[t_r \leq s]$$
$$= \sup_s \sigma[t_r \leq r_s].$$

If s is a stop rule and $s \geq r$, let $s_0 = t_r \wedge r_s$. Then s_0 is a stop rule, $s_0 \geq r$, and $[t_r \leq r_s] = [t_r = s_0] \subseteq \{h: h \in K_{\check{s}_0(h)}\}$. Hence, $\sigma[t_r < \infty] \leq \sup_{s\geq r} \sigma\{h: h \in K_{\check{s}(h)}\}$. But $[t_r < \infty] = \bigcup_n (K_n \cap [r_n \geq r]) \supseteq A_r$, and so $\sigma[t_r < \infty] \geq \sigma(A_r)$. Take the infimum over r to complete the proof. □

8. $\mathcal{G}(\sigma)$ Contains the G_δ's.

As shown in Section 3, every strategy defined on the clopen sets has a natural finitely additive extension to an algebra containing the open sets. An obvious question is: What other sets are in the algebra? The next theorem provides some information, as does Section 10, but a fully satisfactory answer remains to be given.

Theorem 1.

Let σ be a strategy. Then $\mathcal{G}(\sigma)$ contains all sets which are countable intersections of open sets.

The proof uses the following two lemmas.

Lemma 1.

Let $O_1 \supseteq O_2 \supseteq \ldots$ be a sequence of open sets in H. Let $t_1, t_2, \ldots$ be a sequence of incomplete stop rules satisfying:

$$O_n = [t_n < \infty], \quad n = 1,2,3,\ldots.$$

Set

$$K_i = \bigcup_j \, [t_j = i], \; i = 1,2,\ldots.$$

Then

$$(*) \qquad (\bigcap_n O_n) \cap [\limsup t_n = +\infty] = [K_n \text{ occurs i.o.}] \; .$$

Proof:

Suppose $h \in [K_n$ occurs i.o.]. Then there exists $n_1 < n_2 < \ldots$, and $j_1, j_2, j_3, \ldots$ such that $t_{j_k}(h) = n_k$ for all $k \in N$. Now all the j_k's are distinct; since, if $j_k = j_\ell$ then $n_k = t_{j_k}(h) = t_{j_\ell}(h) = n_\ell$. This implies that h belongs to the left hand side of $(*)$.

Next, suppose h is a member of the left hand side of $(*)$. Then there exist positive integers $m_1, m_2, m_3, \ldots$ such that $t_n(h) = m_n$, all n, and $\limsup m_n = +\infty$. The latter implies that the set

$M = \{m_1, m_2, \ldots\}$ is infinite. But $h \in K_j$ for all $j \in M$. □

Lemma 2.

Suppose, in addition to the hypothesis of Lemma 1, that the set $G = \bigcap_n O_n$ has an empty interior. Then $G \subseteq [\limsup t_n = +\infty]$.

Proof:

Suppose, by way of contradiction, that $h \in G$ and $t_n(h) \leq b$ for all n, where b is a fixed positive integer. Then G contains all $\hat{h} \in H$ such that

$$\hat{h}_i = h_i, \quad \text{for } i = 1, \ldots, b.$$

This follows from the fact that each t_n is an incomplete stop rule and $t_n(h) \leq b$. The contradiction is established. □

Proof of Theorem 1.

The lemmas together with Theorem 7.1 imply that $\mathcal{G}(\sigma)$ contains every G_δ (i.e., countable intersection of open sets) with empty interior.

Let G be an arbitrary G_δ and let I be its interior. Then I^c, being a closed subset of the metrizable space H, is also a G_δ. Thus $G \cap I^c$ is a G_δ with empty interior and so is in $\mathcal{G}(\sigma)$. Also, the open set I is in $\mathcal{G}(\sigma)$. Hence, $G = (G \cap I^c) \cup I$ is also. □

9. The Borel-Cantelli Lemmas and a Strong Law of Large Numbers.

The question which is tentatively raised in this section is whether the classical strong limit theorems continue to hold in a finitely additive theory and, if so, in what form. The "Fatou equation" of Section 6 can, for example, be considered as the analogue of Fatou's Lemma. Theorems 1 and 2 below are finitely additive versions of the Borel-Cantelli lemmas. Theorem 3 corresponds to the strong law for independent, uniformly bounded variables.

For the first two theorems, assume the same setting as in paragraph 2 of Section 5.

Theorem 1. Theorem 1

Suppose that for $n = 1, 2, \ldots$ and $h \in H$, $\sigma[q_n(h)](K_{n+1}q_n(h)) \leq \alpha_{n+1}$. If $\Sigma\, \alpha_n < \infty$, then $\sigma[K_n \text{ i.o.}] = 0$.

Proof:

Let $n \in N$. Notice that $\sigma(K^c_{n+1}) = \int \sigma[q_n(h)](K^c_{n+1}q_n(h))d\sigma(h) \geq 1 - \alpha_{n+1}$ and, for $k \in N$ and $h \in H$, $\sigma[q_{n+k}(h)](K^c_{n+k+1}q_{n+k}(h)) \geq 1 - \alpha_{n+k+1}$. By Theorem 5.1, $\sigma(\bigcap_{i>n} K^c_i) \geq \prod_{i>n} (1 - \alpha_i) \geq 1 - \sum_{i>n} \alpha_i \to 1$ as $n \to \infty$. (The second inequality uses the elementary fact that $\prod p_i \geq 1 - \sum p_i$ for numbers p_i such that $0 \leq p_i \leq 1$.) Since, for all n, $[K_m \text{ i.o.}] \subseteq \bigcup_{i>n} K_i = (\bigcap_{i>n} K^c_i)^c$, the proof is complete. □

In the conventional theory, the result corresponding to the previous theorem states that, for arbitrary events A_n, if $\Sigma\, P(A_n) < \infty$, then $P[A_n \text{ i.o.}] = 0$. The same is not true here as the following example shows.

Example.

Let X and σ be as in the example of Section 5. Let $K_n = \{h | h_n \leq n\}$, for $n \in N$. Then $\sigma(K_n) = 0$ for all n, but $\sigma[K_n \text{ i.o.}] = 1$ as can be seen by applying Theorem 4.1.

Corollary 1.

Let $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ be an independent strategy on H as defined in Section 5. Let $i_1, i_2, \ldots$ be a sequence of positive integers and suppose

$$A_n \subseteq X^{i_n} \quad \text{for all } n.$$

Set

$$r_1 = 1;\ r_n = i_1 + \ldots + i_{n-1} + 1,\ n \geq 2;$$

$$s_n = i_1 + \ldots + i_n,\ n \geq 1; \text{ and}$$

$$K_n = \{h\colon (h_{r_n}, \ldots, h_{s_n}) \in A_n\},\ n \geq 1.$$

If $\Sigma\, \sigma(K_n) < \infty$, then $\sigma[K_n \text{ i.o.}] = 1$.

Proof:

For each n, K_n is determined by time r_n and, for all h, $\sigma[q_n(h)](K_{n+1}q_n(h)) = \sigma(K_{n+1})$. Now use Theorem 1. □

A reader mainly interested in the strong law can skip the next theorem and its corollary.

Theorem 2.

Suppose that for $n = 1, 2, \ldots$ and $h \in H$, $\sigma[q_n(h)](K_{n+1}q_n(h)) \geq \alpha_{n+1}$. If $\Sigma\, \alpha_n = \infty$, then $\sigma[K_n \text{ i.o.}] = 1$.

Proof:

Let s be a stop rule. For $h \in H$, let $\hat{s}(h) = r_n(h)$ if $s(h) = n$. Then $\hat{s} \geq s$ and it is easy to check that $\hat{s}$ is a stop rule. By Theorem 7.2, it suffices to show $\sigma(A_{\hat{s}}) = 1$ where $A_{\hat{s}} = \{h\colon h \in \bigcup_{n \geq \hat{s}(h)} K_n\}$.

Let $h \in H$ and suppose $s(h) = n$. Then

$$\begin{aligned}\sigma[p_{\hat{s}}(h)](A_{\hat{s}}p_{\hat{s}}(h)) &= \sigma[q_n(h)](\bigcup_{i \geq n} K_i q_n(h)) \\ &= 1 - \sigma[q_n(h)](\bigcap_{i \geq n} K_i^c q_n(h)) \\ &\geq 1 - \prod_{i \geq n} (1-\alpha_i).\end{aligned}$$

The inequality follows from an application of Theorem 5.1 to the strategy $\sigma[q_n(h)]$ and clopen sets $\{K_i^c q_n(h)\}_{i\geq n}$. Also

$$\prod_{i\geq n} (1-\alpha_i) \leq \prod_{i=n}^{m} (1-\alpha_i) \leq \exp(-\sum_{i=n}^{m} \alpha_i) \to 0 \quad \text{as} \quad m \to \infty,$$

and, hence, $\sigma[p_g(h)](A_g p_g(h)) = 1$ for all h. Now apply Corollary 4.1, to get $\sigma(A_g) = 1$. □

Corollary 2.

Let σ and the K_n be as in Corollary 1. If $\Sigma\, \sigma(K_n) = \infty$, then $\sigma[K_n \text{ i.o.}] = 1$.

Proof:

Similar to that of Corollary 1. □

We are grateful to David Freedman for pointing out that conventional methods now suffice to prove the next result. The particular proof given is similar to that of Theorem 5.1.2 in [1]. In the sequel, σg is used as a brief notation for $\int g d\sigma$.

Theorem 3.

Let σ be an independent strategy on H and, for $n = 1, 2, \ldots$, let Y_n be a real-valued function on H which depends only on the n^{th} coordinate. Assume that $|Y_n(h)| \leq 1$ for all n and h, and that $\sigma Y_n = 0$ for all n. Then the set $\{h: \frac{1}{n} \Sigma_{i=1}^{n} Y_i(h) \to 0\}$ is in $\mathcal{G}(\sigma)$ and has σ-measure one.

Proof:

Let $S_0 = 0$, $S_n = Y_1 + \ldots + Y_n$, and $T_n = S_{n!} - S_{(n-1)!}$ for $n = 1,2,\ldots.$ Notice that $\sigma(S_n^2) \leq n$ and $\sigma(T_n^2) \leq \sigma(S_{n!}^2) \leq n!$ for all n. The proof of Theorem 3 is in four steps.

Step 1. $\sigma[\frac{T_n}{n!} \to 0] = 1$.

To see this, let $K_n = [\frac{|T_n|}{n!} \geq \frac{1}{n}]$. Then apply Chebyshev's inequality, which clearly holds for finitely additive measures, to get

$$\sigma(K_n) = \sigma[\frac{T_n^2}{(n!)^2} \geq \frac{1}{n^2}] \leq \frac{n^2}{n!}.$$

Thus $\Sigma\,\sigma(K_n) < \infty$ and, since the K_n are defined in terms of disjoint sets of coordinates, Corollary 1 applies to give $\sigma[K_n \text{ i.o.}] = 0$. Since

$$\left[\frac{T_n}{n!} \to 0\right] \supseteq [K_n \text{ i.o.}]^c,$$

step 1 is complete.

Step 2. $\sigma\left[\frac{S_{n!}}{n!} \to 0\right] = 1.$

Notice that

$$\frac{|S_{n!}|}{n!} \le \frac{|S_{n!} - T_n|}{n!} + \frac{|T_n|}{n!} \le \frac{(n-1)!}{n!} + \frac{|T_n|}{n!}$$

and, hence,

$$\left[\frac{S_{n!}}{n!} \to 0\right] \supseteq \left[\frac{T_n}{n!} \to 0\right].$$

Step 3. For $n = 1, 2, \ldots$, set $D_n = \max\{|S_k - S_{n!}| : n! < k \le (n+1)!\}$. Then $\sigma\left[\frac{D_n}{n!} \to 0\right] = 1$.

To check this step, first notice that, for all n,

$$\sigma(S_n^4) = \sum_{i=1}^{n} \sigma(Y_i^4) + 6 \sum_{1\le i<j\le n} \sigma(Y_i^2)\sigma(Y_j^2) \le 3n^2.$$

Clearly, the same inequality

holds if S_n is replaced by the sum of any n distinct Y_i's. Furthermore,

$$D_n^4 \le \sum_{k=n!+1}^{(n+1)!} |S_k - S_{n!}|^4$$

and, hence,

$$\sigma(D_n^4) \le 3 \sum_{k=1}^{(n+1)!-n!} k^2 \le 3[(n+1)! - n!]^3 \le 3[(n+1)!]^3.$$

The completion of this step is similar to step 1. For each n, let $L_n = \left[\frac{D_n}{n!} \ge \frac{1}{4\sqrt{n}}\right]$. By Chebyshev,

$$\sigma(L_n) = \sigma\left[\frac{D_n^4}{(n!)^4} \ge \frac{1}{n}\right] \le \frac{3n(n+1)^3}{n!}.$$

Since $\Sigma\,\sigma(L_n) < \infty$, $\sigma[L_n \text{ i.o.}] = 0$. The step is finished because

$$\left[\frac{D_n}{n!} \to 0\right] \supseteq [L_n \text{ i.o.}]^c .$$

Step 4. $\sigma[\frac{S_n}{n} \to 0] = 1.$

For each n, let $m = m(n)$ be that integer such that $m! < n \leq (m+1)!$.

Then

$$\frac{|S_n|}{n} \leq \frac{D_m + |S_{m!}|}{m!} .$$

Thus

$$[\frac{S_n}{n} \to 0] \supseteq [\frac{D_n}{n!} \to 0] \cap [\frac{S_{n!}}{n!} \to 0] .$$

Step 4 and the proof of the theorem are now complete. □

10. Relation to Countably Additive Theory.

If a strategy σ satisfies conventional measurability and countable additivity assumptions, then standard countably additive extension theorems can be applied. It is shown below that the present finitely additive extension is consistent with the conventional one and assigns measure to as many sets. Thus the finitely additive probability theorems of previous sections are, in a sense, extensions of the conventional theory.

Let $\mathcal{B}$ be a sigma-field of subsets of X and let $\mathcal{B}^{\infty} = \mathcal{B} \times \mathcal{B} \times \ldots$ be the product sigma-field of subsets of H. It is assumed in this section that σ is a measurable strategy with respect to $\mathcal{B}$. That is, σ is assumed to satisfy

(i) σ_0 is countably additive when restricted to $\mathcal{B}$ and, for every finite sequence $(x_1, \ldots, x_n)$, $\sigma(x_1, \ldots, x_n)$ is countably additive when restricted to $\mathcal{B}$,

(ii) for every n and every $B \in \mathcal{B}$, $\sigma(x_1, \ldots, x_n)(B)$ is a jointly measurable function of $(x_1, \ldots, x_n)$.

Then, as is well-known, there is a unique countably additive probability measure ν on $\mathcal{B}^{\infty}$ such that $\nu(A) = \sigma(A)$ for every cylinder set A; that is, for every set A of the form $B_1 \times B_2 \times \ldots$ where each $B_i \in \mathcal{B}$ and $B_i = X$ for all but finitely many i. Let $\mathcal{C}$ be the completion of $\mathcal{B}^{\infty}$ under ν.

Theorem 1.

If σ is a measurable strategy with respect to $\mathcal{B}$, then $\mathcal{A}(\sigma)$ contains $\mathcal{C}$ and σ agrees with ν on $\mathcal{C}$.

The proof is given in several rather technical lemmas. The heart of the argument is Lemma 2.

Lemma 1.

Let K be a clopen set and let $K \in \mathcal{B}^{\infty}$. Then $\sigma(K) = \nu(K)$.

Proof:

The proof is by induction on the structure of K and is presented in detail in Section 2 of [4]. □

Lemma 2.

Let t be a $\mathcal{B}^{\infty}$-measurable incomplete stop rule. Then $\sigma[t < \infty] = \nu[t < \infty]$.

Proof:

Notice that

$$\begin{aligned}\sigma[t < \infty] &= \sup\{\sigma[t \leq s]: s \text{ a stop rule}\} \text{ (by Corollary 3.2)}\\ &\geq \sup\{\sigma[t \leq n]: n \text{ a positive integer}\}\\ &= \sup\{\nu[t \leq n]: n \text{ a positive integer}\} \text{ (by the previous lemma)}\\ &= \nu[t < \infty].\end{aligned}$$

The final equation above uses the countable additivity of ν on $\mathcal{B}^{\infty}$.

To complete the proof it suffices to show that, for every stop rule s,

$$\sigma[t \leq s] \leq \sup_n \sigma[t \leq n] \ . \tag{10.1}$$

The proof of (10.1) is by induction on the structure of s. If s is constant, (10.1) is clear. It remains to check the inductive step.

Recall that

$$s[x](h) = s(xh) - 1,$$

and set

$$t[x](h) = t(xh) - 1.$$

Notice that, for each x, $s[x]$ is either a stop rule or identically equal to zero. Also, $s[x]$ has smaller structure than that of s if the structure of s is larger than zero. Similarly, $t[x]$ is either a $\mathcal{B}^{\infty}$-measurable incomplete stop rule or identically zero. Finally, the conditional strategy $\sigma[x]$ is measurable, for each x, because σ is. Now compute

$$(10.2) \qquad \sigma[t \leq s] = \int \sigma[x]([t \leq s]x)d\sigma_0(x)$$

$$= \int \sigma[x][t[x] \leq s[x]]d\sigma_0(x)$$

$$\leq \int \sup_n \sigma[x][t[x] \leq n]d\sigma_0(x).$$

The inequality follows from the inductive assumption.

Let $\epsilon > 0$. For $x \in X$, define

$$N(x) = \min\{k : (\sigma[x][t[x] \leq k]) \geq (\sup_n \sigma[x][t[x] \leq n]) - \epsilon\},$$

and let $M(h) = N(h_1) + 1$ for $h \in H$. Then, by (10.2),

$$(10.3) \qquad \sigma[t \leq s] \leq \int \sigma[x][t[x] \leq N(x)]d\sigma_0(x) + \epsilon$$

$$= \int \sigma[x]([t \leq M]x)d\sigma_0(x) + \epsilon$$

$$= \sigma[t \leq M] + \epsilon$$

$$= \nu[t \leq M] + \epsilon .$$

The last step, which follows from Lemma 1, requires that M be $\mathcal{B}^\infty$-measurable. This will follow easily from the $\mathcal{B}$-measurability of the function

$$x \to \sigma[x][t[x] \leq n], \quad x \in X.$$

This has the form $x \to \sigma[x]Ax$, where A is $\mathcal{B}^\infty$-measurable and has finite structure. The quantity $\sigma[x]Ax$ can be evaluated in a natural way as an iterated integral (see, for example, formula 2.6.1 in [2]) involving finitely additive extensions of the countably additive $\sigma(p)$'s. A little reflection shows that the iterated integral has the same value as the usual Lebesgue integral. The $\mathcal{B}^\infty$-measurability of $x \to \sigma[x]Ax$ then follows by the standard arguments.

Since M is $\mathcal{B}^{\infty}$-measurable and ν is countably additive, there exists an integer n such that $\nu[M \leq n] \geq 1 - \epsilon$. So, by (10.3), $\sigma[t \leq s] \leq \nu[t \leq M] + \epsilon \leq \nu[t \leq n] + 2\epsilon = \sigma[t \leq n] + 2\epsilon$. The last equation is by Lemma 1. Since ϵ was arbitrary, (10.1) is now proved. $\square$

Let $\mathfrak{T}$ be the collection of all $\mathcal{B}^{\infty}$-measurable incomplete stop rules t. For $A \subseteq H$, define

$$\nu^{*}(A) = \inf\{\nu[t < \infty]: t \in \mathfrak{T}, A \subseteq [t < \infty]\},$$

and

$$\nu_{*}(A) = \sup\{\nu[t = \infty]: t \in \mathfrak{T}, A \supseteq [t = \infty]\}.$$

Notice $\nu_{*}(A) = 1 - \nu^{*}(A^{c})$.

Let

$$\mathcal{C}' = \{A \subseteq H: \nu^{*}(A) = \nu_{*}(A)\}.$$

Lemma 3.

The collections $\mathcal{C}$ and $\mathcal{C}'$ coincide. Also, ν^{*} restricted to $\mathcal{C}$ is the completion of ν and, in particular, ν^{*} is countably additive on $\mathcal{C}$.

Proof:

First notice that $\mathcal{C}'$ is a sigma-field. To see this check in order that $\mathcal{C}'$ is closed under the taking of complements, finite unions, and countable increasing unions.

Now let A be a cylinder set in $\mathcal{B}^{\infty}$. Then there is an $n \in N$ and a set $B \subseteq X^{n}$ such that $A = \{(x_1, \ldots, x_n, \ldots) : (x_1, \ldots, x_n) \in B\}$. Define $t(h) = \infty$ or n according as $h \notin A$ or $h \in A$; and $v(h) = \infty$ or n according as $h \in A$ or $h \notin A$. Then $t, v \in \mathfrak{T}$ and $[t < \infty] = A = [v = \infty]$. Thus $A \in \mathcal{C}'$ and $\mathcal{C}' \supseteq \mathcal{B}^{\infty}$.

To see $\mathcal{C}' \subseteq \mathcal{C}$, let $A \in \mathcal{C}'$. Write O for sets of the form $[t < \infty]$ and C for sets of the form $[t = \infty]$ when $t \in \mathfrak{T}$. Then there exist sets

O_n and C_n for $n \in N$, such that the O_n's are decreasing, the C_n's are increasing, $O_n \supseteq A \supseteq C_n$, $\nu(O_n) \to \nu^*(A)$, and $\nu(O_n - C_n) \to 0$. Then $\cup C_n \subseteq A$, $A - \cup C_n \subseteq \cap O_n - \cup C_n$, and $\nu(\cap O_n - \cup C_n) = 0$. Thus A differs from $\cup C_n$ by a subset of a $\mathcal{B}^\infty$ set which is ν-null and, hence, $A \in \mathcal{C}$. Notice also that $\nu^*(A) = \nu(\cup C_n)$. Hence, ν^* agrees with the completion of ν on $\mathcal{C}'$. But $\mathcal{C}'$ is clearly complete for ν^* and so is complete for ν. Therefore $\mathcal{C} = \mathcal{C}'$. □

The next lemma finishes the proof of Theorem 1.

Lemma 4.

For every $A \subseteq H$,

$$\nu^*(A) \geq \sigma^*(A) \geq \sigma_*(A) \geq \nu_*(A).$$

Hence, $\mathcal{C}(\sigma) \supseteq \mathcal{C}$ and σ^* agrees with ν^* on $\mathcal{C}$.

Proof:

Easy, but it requires Lemma 2. □

Two brief remarks conclude this section.

Suppose X is finite or countable and $\mathcal{B}$ is the set of all subsets of X. Then every incomplete stop rule is $\mathcal{B}^\infty$-measurable. Hence, $\sigma^* = \nu^*$ and $\mathcal{C}(\sigma)$ is just the usual completion of $\mathcal{B}^\infty$ under ν. In particular, the usual examples of non-measurable sets give examples of sets not in $\mathcal{C}(\sigma)$.

Finally, $\mathcal{C}(\sigma)$ is sometimes strictly larger than $\mathcal{C}$, since $\mathcal{C}(\sigma)$ always contains all clopen sets and it can easily happen that some clopen sets are not measurable.

REFERENCES

[1] Chung, Kai Lai (1968). A course in probability theory. Harcourt, Brace, & World Inc., New York.

[2] Dubins, Lester E. and Savage, Leonard J. (1965). How to gamble if you must: inequalities for stochastic processes. McGraw-Hill, Inc., New York.

[3] Sudderth, William D. (1971). A "Fatou equation" for randomly stopped variables. Ann. Math. Statist. 42 2143-2146.

[4] Sudderth, William D. (1971). On measurable gambling problems. Ann. Math. Statist. 42 260-269.